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1998 J. Phys. A: Math. Gen. 31 9483

(http://iopscience.iop.org/0305-4470/31/47/009)

J. Phys. A: Math. Gen. **31** (1998) 9483–9492. Printed in the UK PII: S0305-4470(98)91995-3

Solitons from dressing in an algebraic approach to the constrained KP heirachy

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Received 5 March 1998, in final form 23 June 1998

Abstract. The algebraic matrix hierarchy approach based on affine Lie $sl(n)$ algebras leads to a variety of $1 + 1$ soliton equations. By varying the rank of the underlying $sl(n)$ algebra as well as its gradation in the affine setting, one encompasses the set of the soliton equations of the constrained KP hierarchy.

The soliton solutions are then obtained as elements of the orbits of the dressing transformations constructed in terms of representations of the vertex operators of the affine $sl(n)$ algebras realized in the unconventional gradations. Such soliton solutions exhibit non-trivial dependence on the KdV (odd) time flows and KP (odd and even) time flows which distinguishes them from the conventional structure of the Darboux–Bäcklund–Wronskian solutions of the constrained KP hierarchy.

1. Introduction. The algebraic cKP model

A large class of $1 + 1$ soliton equations belongs to the so-called constrained KP (cKP) hierarchy. Some of the most prominent members of this group are the KdV and the nonlinear Schrödinger equations of the AKNS model. The cKP evolution equations possess the familiar Lax pair representations with generally pseudo-differential Lax operators which emerge naturally as reductions of the complete KP hierarchy Lax operators [1]. Conventionally, the cKP hierarchy is obtained from the KP hierarchy by a process of reduction involving the so-called eigenfunctions. The eigenfunctions appear in the constraint relations introducing a functional dependence between initially infinitely many coefficients of the KP Lax operator. This scheme results in the pseudo-differential cKP Lax operator of the type $\mathcal{L} = L_{K+1} + \sum_{i=1}^{M} \Phi_i \partial^{-1} \Psi_i$, where L_{K+1} is the differential operator of $(K + 1)$th order, while Φ_i, Ψ_i are the eigenfunctions of $\mathcal{L}$. In general, $\mathcal{L}$ possesses a finite number of coefficients which enter the soliton equations and depend on all $(t_1, t_2, t_3, \ldots)$ isospectral time flows of the KP hierarchy. In the special $M = 0$ case in which the cKP Lax operator is a purely differential operator $\mathcal{L} = L_{K+1}$ we encounter dependence on only some of the original time flows of the KP hierarchy. The simplest example ($K = 1$, $M = 0$) is the KdV hierarchy with only odd time flows present.

The soliton solutions for the cKP models have been found in [2] for the arbitrary K and $M = 1$ case using the Darboux–Bäcklund technique. Generalization to an arbitrary M is simple and was given in [3] (see also [4]). These solutions appeared in the Wronskian form in terms of the eigenfunctions of the 'undressed' $\mathcal{L} = \partial^{K+1}$ Lax operator.

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Here, we will present an alternative algebraic viewpoint of the constrained KP hierarchy. In this setting the algebraic dressing methods will provide new soliton solutions which appear to differ from the conventional form of the Darboux–Bäcklund–Wronskian solutions due to a non-trivial mixing of the KdV-like versus KP time flows. This will be shown explicitly in the example characterized by $K = M = 1$.

In an algebraic approach to the constrained KP hierarchy [5] the soliton evolution equations emerge as integrability conditions of the following matrix eigenvalue problem:

$$L\Psi = 0 \qquad L \equiv (D - A - E) \qquad D \equiv I\frac{\partial}{\partial x} \tag{1}$$

with the matrix Lax operator L belonging to Kac–Moody algebra $\hat{\mathcal{G}} = \widehat{sl}(M + K + 1)$. The integrable hierarchy is determined by the choice of gradation of $\hat{\mathcal{G}}$. By varying the Kac–Moody algebras together with their gradations one is able to reproduce from the matrix hierarchy of equation (1) the nonlinear evolution equations of the cKP hierarchy.

We will be working with a simple setting in which the matrix E in equation (1) has gradation 1 with respect to gradation specified by the vector [6]:

$$\boldsymbol{s} = (1, \underbrace{0, \ldots, 0}_{M}, \underbrace{1, \ldots, 1}_{K}). \tag{2}$$

We call this gradation an intermediate gradation as it interpolates between the principal $\boldsymbol{s}_{\text{principal}} = (1, 1, \ldots, 1)$ and the homogeneous one $\boldsymbol{s}_{\text{homogeneous}} = (1, 0, \ldots, 0)$. As is well known the Wilson–Drinfeld–Sokolov [7–11] procedure gives, respectively, the (m-)KdV [9] and AKNS [12, 13] hierarchies in these two limits.

Alternatively, the gradation $\boldsymbol{s}$ can be specified by an operator:

$$Q_{\boldsymbol{s}} \equiv \sum_{a=1}^{K} \lambda_{M+a} \cdot H^0 + (K+1)d. \tag{3}$$

Here λ_j are the fundamental weights and d is the standard loop algebra derivation. Correspondingly, E stands for

$$E = \sum_{a=1}^{K} E^{(0)}_{\alpha_{M+a}} + E^{(1)}_{-(\alpha_{M+1}+\cdots+\alpha_{M+K})}. \tag{4}$$

It is a non-regular (for $M > 0$) and semisimple element of $\hat{\mathcal{G}}$.

The matrix A in equation (1) contains the dynamical variables of the model. A is such that it has gradation zero and is parametrized in terms of the dynamical variables q_i, r_i, U_a and ν as follows:

$$A = \sum_{i=1}^{M} (q_i P_i + r_i P_{-i}) + \sum_{a=1}^{K} U_{M+a}(\alpha_{M+a} \cdot H^{(0)}) + \nu\hat{c} \tag{5}$$

where $P_{\pm i} = E^{(0)}_{\pm(\alpha_i+\alpha_{i+1}+\cdots+\alpha_M)}$, $i = 1, 2, \ldots, M$, and $\hat{c}$ is a central element of $\hat{\mathcal{G}}$.

2. The Heisenberg subalgebra and the vertex operator

For a regular element E in the conventional Drinfeld–Sokolov approach the isospectral flows are associated with the Heisenberg subalgebra which can be identified with $\mathrm{Ker}(\mathrm{ad}\, E)$. Here, due to the non-regularity of E, the Heisenberg algebra is associated to the centre of $\mathrm{Ker}(\mathrm{ad}\, E)$. It consists of the following three separate sets of operators.

(1) *A homogeneous part of* $\widehat{sl}(M)$, *for* $i = 1, 2, \ldots, M-1$

$$\mathcal{K}_i^{(n)} = \frac{\sum_{p=1}^{i} p\alpha_p \cdot H^{(n)}}{N_i} \qquad N_i \equiv \sqrt{i(i+1)}. \tag{6}$$

(2) *A principal part of* $\widehat{sl}(K+1)$, *for* $a = 1, 2, \ldots, K$

$$\mathcal{A}^a_{a+n(K+1)} = \sum_{i=1}^{K+1-a} E^{(n)}_{\alpha_{i+M}+\alpha_{i+M+1}+\cdots+\alpha_{i+M+a-1}} + \sum_{i=1}^{a} E^{(n+1)}_{-(\alpha_{i+M}+\alpha_{i+M+1}+\cdots+\alpha_{i+M+K-a})}. \tag{7}$$

(3) *'A border term'*

$$\mathcal{A}^0_{n(K+1)} = \sqrt{\frac{M+K+1}{M}}\lambda_M \cdot H^{(n)} - \frac{K}{2}\sqrt{\frac{M}{M+K+1}}\hat{c}\delta_{n,0}. \tag{8}$$

These relations provide a parametrization of the Heisenberg subalgebra in terms of elements:

$$b_{N,a} \equiv \mathcal{A}^a_{N=a+n(K+1)} \qquad b_{N,0} \equiv \mathcal{A}^0_{N=n(K+1)} \qquad b_{N,i} \equiv \mathcal{K}_i^{(N)} \tag{9}$$

where $a = 1, 2, \ldots, K$, $i = 1, 2, \ldots, M-1$. The Heisenberg subalgebra elements from (9) enter the oscillator algebra relations (we put $c = 1$):

$$[b_{N,a}, b_{N',b}] = N\delta_{N+N'}\delta_{a,K+1-b} \qquad a, b = 1, 2, \ldots, K \tag{10}$$

$$[b_{N,0}, b_{N',0}] = N\delta_{N+N'} \tag{11}$$

$$[b_{N,i}, b_{N',j}] = N\delta_{N+N'}\delta_{ij} \qquad i, j = 1, 2, \ldots, M-1. \tag{12}$$

Next define the Fubini–Veneziano operators:

$$Q_{1\leqslant i\leqslant M-1}(z) = \mathrm{i}\sum_{n=1}^{\infty} \frac{\mathcal{K}_i^{(n)} z^{-n}}{n} \qquad Q_M(z) = \mathrm{i}\sum_{n=1}^{\infty}\frac{\mathcal{A}^0_{n(K+1)} z^{-n(K+1)}}{n(K+1)} \tag{13}$$

$$Q_{M+a}(z) = \mathrm{i}\sum_{n=0}^{\infty}\frac{\mathcal{A}^a_{a+n(K+1)} z^{a+n(K+1)}}{a+n(K+1)} \qquad a = 1, 2, \ldots, K. \tag{14}$$

The corresponding conjugated Fubini–Veneziano operators $Q^\dagger(z)$ are obtained from (13), (14) by taking into consideration the rules $\mathcal{K}^{(n)}{}^\dagger_i = \mathcal{K}_i^{(-n)}$, $(\mathcal{A}^0_{n(K+1)})^\dagger = \mathcal{A}^0_{-n(K+1)}$, $(\mathcal{A}^a_{a+n(K+1)})^\dagger = \mathcal{A}^a_{a-n(K+1)}$, as well as $z^\dagger = z^{-1}$.

In [14] we found the step operators of $sl(M+K+1)$ associated with the Cartan subalgebra defined by the Heisenberg subalgebra (9). That in turn enabled us to find the corresponding simple root structure for $sl(M+K+1)$ with intermediate grading.

Knowledge of roots and the Fubini–Veneziano operators is all that is needed to write a compact expression for the general vertex operator in the normal ordered form:

$$V^{\alpha}(z) \equiv z^{\frac{1}{2}\sum_{j=1}^{M}(\alpha^j)^2} \exp(\mathrm{i}\boldsymbol{\alpha}^* \cdot \boldsymbol{Q}^\dagger(z)) \exp(\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{q}) \exp(\boldsymbol{\alpha}\cdot\boldsymbol{p}\ln z) \exp(\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{Q}(z)) \tag{15}$$

with the $(M+K)$-component root vector $\boldsymbol{\alpha}$ described in [14] and $(M+K)$-component vector $\boldsymbol{Q}$ having components described in (13), (14). The zero-mode vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ only have first M components different from zero according to $(\boldsymbol{p})_i = p_i\theta(M-i)$ and $(\boldsymbol{q})_i = q_i\theta(M-i)$. They satisfy relations $[p_i q_j] = -\mathrm{i}\delta_{ij}$. Furthermore, p_M is equal to $\mathcal{A}^0_{n=0}$ from expression (8).

An explicit example of the $sl(3)$ vertex will be given in section 4.

3. The dressing technique and the tau-function

The dressing technique [15] deals with reproducing of the non-trivial part $E + A$ of the Lax matrix operator from equation (1) by the gauge transformations involving generators of positive and negative gradings applied to the semisimple element E:

$$E + A = \Theta E \Theta^{-1} + (\partial_x \Theta)\Theta^{-1} \tag{16}$$

$$E + A = (B^{-1}\Gamma)E(\Gamma^{-1}B) + (\partial_x B^{-1}\Gamma)(\Gamma^{-1}B) \tag{17}$$

where $B^{-1}\Gamma$ contains positive terms and Θ is an expansion in the terms of negative grading such that $\Theta = \exp(\sum_{l<0}\theta^{(l)}) = 1 + \theta^{(-1)} + \cdots$. From expressions (16) and (17) we obtain two alternative formulae for the same term A of grade 0:

$$A = -[E\theta^{(-1)}] \qquad \text{or} \qquad A = -B^{-1}(\partial_x B). \tag{18}$$

The term $\theta^{(-1)}$ of grade -1 can be expanded as

$$\theta^{(-1)} = \sum_{a=M+1}^{M+K} \theta_a^{(-1)} E^{(0)}_{-\alpha_a} + \theta_\psi^{(-1)} E^{(-1)}_{\alpha_{M+1}+\cdots+\alpha_{M+K}} + \sum_{l=1}^{M} \theta_l^{(-1)} E^{(0)}_{-(\alpha_l+\cdots+\alpha_{M+1})} + \sum_{l=1}^{M} \bar{\theta}_l^{(-1)} E^{(-1)}_{\alpha_l+\cdots+\alpha_{M+K}} \tag{19}$$

where we included all possible terms of grade -1 according to (3).

Therefore

$$\begin{aligned}[E\theta,^{(-1)}] = {} & \sum_{a=M+1}^{M+K} \theta_a^{(-1)}\alpha_a \cdot H^{(0)} + \theta_\psi^{(-1)}(-(\alpha_{M+1} + \cdots + \alpha_{M+K}) \cdot H^{(0)} + c) \\ & + \sum_{l=1}^{M} \theta_l^{(-1)} \epsilon(\alpha_{M+1}, -\alpha_l - \cdots - \alpha_{M+1}) E^{(0)}_{-(\alpha_l+\cdots+\alpha_M)} \\ & + \sum_{l=1}^{M} \bar{\theta}_l^{(-1)} \epsilon(-\alpha_{M+1} - \cdots - \alpha_{M+K}, \alpha_l + \cdots + \alpha_{M+K}) E^{(0)}_{\alpha_l+\cdots+\alpha_M}.\end{aligned} \tag{20}$$

Comparing the last expression with the field content of A, as given by (5), we obtain relations for the expansion parameters used in (19):

$$\begin{aligned} &\nu = -\theta_\psi^{(-1)} \qquad U_a = -\theta_a^{(-1)} + \theta_\psi^{(-1)} \qquad r_l = -\theta_l^{(-1)} \epsilon(\alpha_{M+1}, -\alpha_l - \cdots - \alpha_{M+1}) \\ &q_l = -\bar{\theta}_l^{(-1)} \epsilon(-\alpha_{M+1} - \cdots - \alpha_{M+K}, \alpha_l + \cdots + \alpha_{M+K}). \end{aligned} \tag{21}$$

We now work with the representation of A as given in equation (18). We split the grade-zero element B in a product $B = B_1 B_2$ with B_1 containing the grade-zero $sl(M)$ elements and

$$B_2 \equiv \exp \sum_{a=1}^{K} \phi_{M+a}\alpha_{M+a} \cdot H^{(0)} + \rho \cdot \hat{c}. \tag{22}$$

Accordingly, equation (18) becomes $A = -B_2^{-1}B_1^{-1}(\partial_x B_1)B_2 - B_2^{-1}(\partial_x B_2)$ and A can be rewritten as

$$A = -\sum_{a=1}^{K} \partial_x \phi_{M+a}\alpha_{M+a} \cdot H^{(0)} - \partial_x \rho \cdot \hat{c} + \mathrm{O}(sl(M)) \tag{23}$$

where $\mathrm{O}(sl(M))$ contains all possible terms belonging to the $sl(M)$ algebra.

Comparison with (5) yields

$$U_{M+a} = -\partial_x \phi_{M+a} \qquad \nu = -\partial_x \rho. \tag{24}$$

We define a family of the first-order differential matrix operators $\mathcal{L}_N = \partial/\partial t_N - A_N$, $N = 1, \ldots$. The hierarchy is then formulated in terms of the zero-curvature equations for the Lax operators

$$[\mathcal{L}_N, \mathcal{L}_M] = 0 \tag{25}$$

expressing commutativity of the higher time flows. The zero-curvature equations imply the pure gauge solutions for the potentials A_N:

$$\mathcal{L}_N = \Psi \frac{\partial}{\partial t_N} \Psi^{-1}. \tag{26}$$

The starting point of the dressing method [15] is the vacuum solution $\nu = U = r = q = 0$. The corresponding $L^{(\mathrm{vac})} = D - E$ matrix Lax operator together with higher flow operators $\mathcal{L}_N$, $N > 1$ for the vacuum solutions are expected to be recovered via (26) from Ψ, which is expressed entirely by the Heisenberg algebra associated with the centre of $\mathrm{Ker}(\mathrm{ad}\, E)$. Explicitly, for our model

$$\Psi = \Psi^{(\mathrm{vac})} = \exp\left(\sum_N t_N b^{(N)} \right) \tag{27}$$

with b_N as given in components in (9) and with the sum in N including all non-negative modes of oscillators appearing in equation (9).

We define the tau-function vectors as

$$|\tau_0\rangle = \Psi^{(\mathrm{vac})} h \Psi^{(\mathrm{vac})\,-1} |\lambda_0\rangle \qquad |\tau_{M+a}\rangle = \Psi^{(\mathrm{vac})} h \Psi^{(\mathrm{vac})\,-1} |\lambda_{M+a}\rangle. \tag{28}$$

They are associated with the constant group element h and the highest-weight vectors $|\lambda_0\rangle$, $|\lambda_{M+a}\rangle$ such that

$$\alpha_{M+a} \cdot H^{(0)} |\lambda_0\rangle = 0 \qquad \alpha_{M+b} \cdot H^{(0)} |\lambda_{M+a}\rangle = \delta_{a,b} |\lambda_{M+a}\rangle \qquad a, b = 1, \ldots, K. \tag{29}$$

Assuming that h allows the 'Gauss' decomposition of $\Psi^{(\mathrm{vac})} h \Psi^{(\mathrm{vac})\,-1}$ in positive, negative and grade-zero elements we get for the tau-function vectors from (28) an alternative expression:

$$\begin{aligned} &|\tau_0\rangle = \Theta^{-1} B^{-1} |\lambda_0\rangle \qquad |\tau_{M+a}\rangle = \Theta^{-1} B^{-1} |\lambda_{M+a}\rangle \\ &\Theta^{-1} = (\Psi^{(\mathrm{vac})} h \Psi^{(\mathrm{vac})\,-1})_- \qquad B^{-1} = (\Psi^{(\mathrm{vac})} h \Psi^{(\mathrm{vac})\,-1})_0 \end{aligned} \tag{30}$$

for $a = 1, \ldots, K$. As before, we make a splitting $B^{-1} = B_2^{-1} B_1^{-1}$ in (30) and notice that $B_1^{-1} |\lambda_{M+a}\rangle = |\lambda_{M+a}\rangle$ for B_1 being an exponential of $sl(M)$ generators. Inserting B_2 from (22) we find

$$|\tau_0\rangle = \Theta^{-1} |\lambda_0\rangle \mathrm{e}^{(-\rho)} \qquad |\tau_{M+a}\rangle = \Theta^{-1} B^{-1} |\lambda_{M+a}\rangle \mathrm{e}^{(-\rho - \phi_{M+a})}. \tag{31}$$

Denote

$$\tau_0^{(0)} \equiv \exp(-\rho) \qquad \tau_{M+a}^{(0)} \equiv \exp(-\rho - \phi_{M+a}). \tag{32}$$

Accordingly, expanding Θ^{-1}, as below equation (17), we find

$$\frac{|\tau_{M+a}\rangle}{\tau_{M+a}^{(0)}} = (1 - \theta^{(-1)} - \cdots) |\lambda_{M+a}\rangle \tag{33}$$

and similarly for $|\tau_0\rangle/\tau_0^{(0)}$. We find by comparing with relation (21) that

$$r_l = \epsilon(\alpha_{M+1}, -\alpha_l - \cdots - \alpha_{M+1})\langle\lambda_{M+1}|E^{(0)}_{\alpha_l+\cdots+\alpha_{M+1}}|\tau_{M+1}\rangle/\tau^{(0)}_{M+1} \tag{34}$$

$$q_l = \epsilon(-\alpha_{M+1} - \cdots - \alpha_{M+K}, \alpha_l + \cdots + \alpha_{M+K})\langle\lambda_0|E^{(1)}_{-(\alpha_l+\cdots+\alpha_{M+K})}|\tau_0\rangle/\tau_0^{(0)} \tag{35}$$

$$U_{M+a} = -\partial_x \ln(\tau_0^{(0)}/\tau^{(0)}_{M+a}) \qquad \nu = -\partial_x \ln(\tau_0^{(0)}). \tag{36}$$

The multisoliton tau-functions are defined in terms of the constant group elements h which are the product of exponentials of eigenvectors of the Heisenberg subalgebra elements

$$h = \mathrm{e}^{F_1}\mathrm{e}^{F_2}\ldots\mathrm{e}^{F_n} \qquad [b_N, F_k] = \omega_N^{(k)} F_k \qquad k = 1, 2, \ldots, n. \tag{37}$$

As seen from equation (37) for such group elements the dependence of the tau-vectors upon the times t_N can be made quite explicit

$$|\tau_a\rangle = \prod_{k=1}^{n} \exp(\mathrm{e}^{\sum_N \omega_N^{(k)} t_N} F_k)|\lambda_a\rangle. \tag{38}$$

The multisoliton solutions are conveniently obtained in terms of representations of the 'vertex operator' type where the corresponding eigenvectors are nilpotent.

4. The $sl(3)$ example: solitons of the Yaijma–Oikawa hierarchy

We apply the above method to the particular case of $sl(3)$ with $M = K = 1$. From equation (9) the surviving elements of the Heisenberg subalgebra are in this case:

$$b^{(2n+1)} \equiv b_{N=2n+1,a=1} = \mathcal{A}^1_{N=1+n\cdot 2} = E^{(n)}_{\alpha_2} + E^{(n+1)}_{-\alpha_2} \tag{39}$$

$$b^{(2n)} \equiv b_{N=2n,0} = \sqrt{3}\lambda_1 \cdot H^{(n)} - \frac{\hat{c}}{2\sqrt{3}}\delta_{n,0} \tag{40}$$

and they satisfy the usual Heisenberg subalgebra $[b^{(k)}, b^{(k')}] = k\delta_{k+k'}$ for both even and odd k.

The structure of eigenvectors of Heisenberg subalgebra facilitates construction of multisoliton solutions according to (37) and (38). In the current example we find that the eigenvectors and their corresponding eigenvalues (in the notation of (37)) are

$$E_{\tilde{\alpha}_1} = \sqrt{2}\sum_{n\in\mathbb{Z}}[z^{-2n}E^{(n)}_{\alpha_1} - z^{-2n-1}E^{(n)}_{\alpha_1+\alpha_2}]$$
$$\omega^{(2n+1)}_{\tilde{\alpha}_1} = z^{2n+1} \qquad \omega^{(2n)}_{\tilde{\alpha}_1} = \sqrt{3}z^{2n} \tag{41}$$

$$E_{\tilde{\alpha}_2} = \sum_{n\in\mathbb{Z}}\left[z^{-2n-1}(E^{(n)}_{\alpha_2} - E^{(n+1)}_{-\alpha_2}) + z^{-2n}\left(\alpha_2 \cdot H^{(n)} - \frac{\hat{c}}{2}\delta_{n,0}\right)\right]$$
$$\omega^{(2n+1)}_{\tilde{\alpha}_2} = -2z^{2n+1} \qquad \omega^{(2n)}_{\tilde{\alpha}_2} = 0 \tag{42}$$

$$E_{\tilde{\alpha}_1+\tilde{\alpha}_2} = \sqrt{2}\sum_{n\in\mathbb{Z}}[z^{-2n}E^{(n)}_{\alpha_1} + z^{-2n-1}E^{(n)}_{\alpha_1+\alpha_2}]$$
$$\omega^{(2n+1)}_{\tilde{\alpha}_1+\tilde{\alpha}_2} = -z^{2n+1} \qquad \omega^{(2n)}_{\tilde{\alpha}_1+\tilde{\alpha}_2} = \sqrt{3}z^{2n}. \tag{43}$$

We now realize the above eigenvectors by the nilpotent vertex operators. The construction involves the Fubini–Veneziano operators defined in terms of the Heisenberg elements as in equations (13), (14):

$$Q_1(z) \equiv \mathrm{i}\sum_{n\in\mathbb{Z}}\frac{b^{(2n+1)}z^{-2n-1}}{2n+1} \qquad Q_2(z) \equiv q - \mathrm{i}p\ln z + \mathrm{i}\sum_{n\neq 0}\frac{b^{(2n)}z^{-2n}}{2n} \tag{44}$$

where the zero-mode momentum $p = b^{(0)} = \sqrt{3}\lambda_1 \cdot H^{(0)} - \hat{c}/2\sqrt{3}$ satisfies $[q, p] = i$. The step operators from (41), (43) are then realized, from the algebra point of view, as vertex operators via:

$$E_{\tilde{\alpha}_1} \leftrightarrow E_{(1,\sqrt{3})}(z) = \sqrt{2}z^{3/2} : \exp(\mathrm{i}Q_1(z) + \mathrm{i}\sqrt{3}Q_2(z)) \tag{45}$$

$$E_{\tilde{\alpha}_2} \leftrightarrow E_{(-2,0)}(z) = -\tfrac{1}{2} : \exp(-2\mathrm{i}Q_1(z)) : \mathrm{e}^{\mathrm{i}\pi p} \tag{46}$$

$$E_{\tilde{\alpha}_1+\tilde{\alpha}_2} \leftrightarrow E_{(-1,\sqrt{3})}(z) = \sqrt{2}z^{3/2} : \exp(-\mathrm{i}Q_1(z) + \mathrm{i}\sqrt{3}Q_2(z)) \tag{47}$$

and similarly for the negative root step operators, with a change of sign of i in exponentials. Care has to be exercised in applying this correspondence within the setting of the Fock space where the vacuum vector is $|\lambda_2\rangle$, since $\langle\lambda_0|E_{\tilde{\alpha}_2}|\lambda_0\rangle = -\frac{1}{2}$ while $\langle\lambda_2|E_{\tilde{\alpha}_2}|\lambda_2\rangle = \frac{1}{2}$, as seen from expression (42). Similar consideration applies for the products of E_{α_i}'s vertex operators such as $E_{(-1,-\sqrt{3})}E_{(-1,\sqrt{3})}$ which produce $E_{(-2,0)}$.

Introduce the notation:

$$V_{c_i,d_i}(z) \equiv z^{d_i^2/2} : \exp(\mathrm{i}c_i Q_1(z) + \mathrm{i}d_i Q_2(z)). \tag{48}$$

It is not difficult to establish the following correlation function:

$$\langle\lambda_\sigma|\Psi^{(\mathrm{vac})}V_{c_1,d_1}(z_1)\ldots V_{c_n,d_n}(z_n)\Psi^{(\mathrm{vac})-1}|\lambda_\sigma\rangle = \delta_{\sum_{j=1}^n d_j,0}\mathrm{e}^{\sum_{j=1}^n \Gamma_{c_j,d_j}(z_j)}$$
$$\times\prod_{j=1}^n z_j^{(-1)^{(\sigma+2)/2}\frac{d_j}{2\sqrt{3}}+\frac{d_j^2}{2}} \prod_{1\leqslant i<j\leqslant n}\left(\frac{z_i - z_j}{z_i + z_j}\right)^{c_i c_j/2}[(z_i - z_j)(z_i + z_j)]^{d_i d_j/2} \tag{49}$$

for $\sigma = 0, 2$ and with

$$\Gamma_{c_j,d_j}(z_j) = \sum_{n=0}^{\infty} c_j t_{2n+1} z_j^{2n+1} + \sum_{n=1}^{\infty} d_j t_{2n} z_j^{2n}. \tag{50}$$

When substituting $V_{c_j,d_j}(z_j)$ by $E_{c_j,d_j}(z_j)$ one encounters extra phases originating from the Klein factor in equation (46) and from the character of the $|\lambda_2\rangle$ vacuum as discussed in equation (47). The latter gives rise to the factor $\exp((\mathrm{i}\pi/2)\sum_{j=1}^n c_j)$ for the λ_2 correlation function as verified on several examples.

Recall that for the problem in hand the Lax matrix operator from (1) with A from (5) and E from (4) specifies to

$$L = D - \begin{pmatrix} 0 & q & 0 \\ r & U_2 & 1 \\ 0 & \lambda & -U_2 \end{pmatrix} - \nu\hat{c} \tag{51}$$

where λ is the usual loop parameter. In terms of the tau-vectors we have from (34)–(36) the following n-soliton representation of the components of the Lax operator

$$r = \frac{1}{\tau_2^{(0)}}\langle\lambda_2|E^{(0)}_{\alpha_1+\alpha_2}|\tau_2\rangle = \frac{1}{\tau_2^{(0)}}\left\langle\lambda_2|E^{(0)}_{\alpha_1+\alpha_2}\Psi^{(\mathrm{vac})}\prod_{j=1}^n(1 + E_{c_j,d_j}(z_j))\Psi^{(\mathrm{vac})-1}|\lambda_2\right\rangle \tag{52}$$

$$q = \frac{1}{\tau_0^{(0)}}\langle\lambda_0|E^{(1)}_{-\alpha_1-\alpha_2}|\tau_0\rangle = \frac{1}{\tau_0^{(0)}}\left\langle\lambda_0|E^{(1)}_{-\alpha_1-\alpha_2}\Psi^{(\mathrm{vac})}\prod_{j=1}^n(1 + E_{c_j,d_j}(z_j))\Psi^{(\mathrm{vac})-1}|\lambda_0\right\rangle \tag{53}$$

$$U_2 = -\partial_x \ln(\tau_0^{(0)}/\tau_2^{(0)}) \qquad \nu = -\partial_x \ln(\tau_0^{(0)}) \tag{54}$$

where

$$\tau_\sigma^{(0)} = \left\langle\lambda_\sigma|\Psi^{(\mathrm{vac})}\prod_{j=1}^n(1 + E_{c_j,d_j}(z_j))\Psi^{(\mathrm{vac})-1}|\lambda_\sigma\right\rangle \qquad \sigma = 0, 2. \tag{55}$$

Using association between the step operators (41)–(43) and the vertex operators (45)–(47) we can rewrite the step operators appearing in (52) and (53) as

$$E^{(0)}_{\alpha_1+\alpha_2} = -\frac{1}{2\mathrm{i}\pi}\int \delta z_0 V_{1,\sqrt{3}}(z_0) \qquad E^{(1)}_{-\alpha_1-\alpha_2} = \frac{1}{2\mathrm{i}\pi}\int \delta z_0 V_{1,-\sqrt{3}}(z_0). \tag{56}$$

We now calculate the zero-curvature equations (25) $[D - E - A\partial_{t_n} - A_n] = 0$ for the first two non-trivial cases of $n = 2, 3$. We expand $A_n = \sum_{i=0}^{n} A_n(i)$ where the index i in the parenthesis equals grading with respect to $Q_s = \lambda_2 \cdot H^0 + 2d$. We choose $A_3(3) = E^{(1)}_{\alpha_2} + E^{(2)}_{-\alpha_2}$ and $A_2(2) = \sqrt{3}\lambda_1 \cdot H^{(1)}$ in order to ensure truncation of the expansion. This method yields for the first non-trivial case ($n = 2$) the evolution equations

$$0 = \partial_t r + \partial_x^2 r + r\partial_x U_2 - qr^2 - U_2^2 r \qquad 0 = \partial_t U_2 + \partial_x(rq) \tag{57}$$

$$0 = \partial_t q - \partial_x^2 q + q\partial_x U_2 + rq^2 + U_2^2 q \tag{58}$$

where we defined for simplicity $t = \sqrt{3}t_2$. The evolution equations for t_3 are

$$0 = \partial_{t_3} U_2 - \tfrac{1}{4}\partial_x^3 U_2 + \tfrac{1}{2}\partial_x U_2^3 + \tfrac{3}{4}\partial_x(r\partial_x q - q\partial_x r) \tag{59}$$

$$\begin{aligned} 0 = \partial_{t_3} r - \partial_x^3 r - \tfrac{3}{2}\partial_x r\partial_x U_2 - \tfrac{3}{4} r\partial_x^2 U_2 + \tfrac{3}{2} rU_2\partial_x U_2 + \tfrac{3}{2}U_2^2\partial_x r \\ + \tfrac{3}{2} qr^2 U_2 + \tfrac{9}{4} rq\partial_x r - \tfrac{3}{4} r^2\partial_x q \end{aligned} \tag{60}$$

$$\begin{aligned} 0 = \partial_{t_3} q - \partial_x^3 q + \tfrac{3}{2}\partial_x q\partial_x U_2 + \tfrac{3}{4} q\partial_x^2 U_2 + \tfrac{3}{2} qU_2\partial_x U_2 + \tfrac{3}{2}U_2^2\partial_x q \\ - \tfrac{3}{2} q^2 rU_2 + \tfrac{9}{4} rq\partial_x q - \tfrac{3}{4} q^2\partial_x r. \end{aligned} \tag{61}$$

These equations follow also from the conventional Sato equations $\partial_{t_n}\mathcal{L} = [(\mathcal{L})_+^{(n/2)}, \mathcal{L}]$ applied to the scalar cKP Lax operator $\mathcal{L} = (\partial - U_2)(\partial + U_2 - q\partial^{-1}r)$. We note here that the simple reduction of the matrix Lax operator from (51) yields the scalar spectral problem $\mathcal{L}_1\chi = \lambda\chi$ with the scalar Lax operator $\mathcal{L}_1 = (\partial + U_2)(\partial - U_2 - r\partial^{-1}q)$. Both scalar Lax operators are related by a conjugation and the Darboux–Bäcklund transformation: $\mathcal{L}_1 = (\partial + U_2)\mathcal{L}^*(\partial + U_2)^{-1}$.

We now present few examples of the soliton solutions (52)–(55) satisfying the above evolution equations.

(1) *Soliton solution for* $n = 1$. With $h = (1 + E_{-2,0}(z_1))$ we recover the standard m-KdV one-soliton configuration with $r = q = 0$ and

$$\tau_0^{(0)} = 1 - \tfrac{1}{2}\mathrm{e}^{(-2xz_1 - 2t_3 z_1^3)} \qquad \tau_2^{(0)} = 1 + \tfrac{1}{2}\mathrm{e}^{(-2xz_1 - 2t_3 z_1^3)}. \tag{62}$$

(2) *Soliton solutions for* $n = 2$. For $h = (1 + E_{-2,0}(z_1))(1 + E_{1,\sqrt{3}}(z_2))$ we find $\tau_0^{(0)}$ and $\tau_2^{(0)}$ as in (62) but now with $q \neq 0$ and equal to

$$q = -\sqrt{2}z_2\mathrm{e}^{(t_3 {z_2}^3 + tz_2^2 + xz_2)}\left(1 + \frac{1}{2}\mathrm{e}^{(-2xz_1 - 2t_3 z_1^3)}\left(\frac{z_1 + z_2}{z_1 - z_2}\right)\right)/\tau_0^{(0)} \tag{63}$$

while $r = 0$. Similarly, for $h = (1 + E_{-2,0}(z_1))(1 + E_{1,-\sqrt{3}}(z_2))$ we find $r \neq 0$ but $q = 0$.

For $h = (1 + E_{1,\sqrt{3}}(z_1))(1 + E_{1,-\sqrt{3}}(z_2))$ we find that both $q \neq 0$ and $r \neq 0$:

$$\tau_\sigma^{(0)} = 1 + (-1)^{(\sigma/2)} 2\frac{z_1^{1+\sigma/2} z_2^{2-\sigma/2}\mathrm{e}^{(xz_1 + tz_1^2 + t_3 z_1^3 + xz_2 - tz_2^2 + t_3 z_2^3)}}{(z_1 - z_2)(z_1 + z_2)^2} \qquad \sigma = 0, 2 \tag{64}$$

$$r = \frac{\sqrt{2}z_2\mathrm{e}^{(-tz_2^2 + xz_2 + t_3 z_2^3)}}{\tau_2^{(0)}} \qquad q = \frac{\sqrt{2}z_1\mathrm{e}^{(tz_1^2 + xz_1 + t_3 z_1^3)}}{\tau_0^{(0)}}. \tag{65}$$

(3) *Soliton solutions for* $n = 3$. As an example we take here $h = (1 + E_{-2,0}(z_1))(1 + E_{1,\sqrt{3}}(z_2))(1 + E_{1,-\sqrt{3}}(z_3))$. We find

$$\tau_\sigma^{(0)} = 1 + (-1)^{(\sigma/2)}\tfrac{1}{2}\mathrm{e}^{(-2xz_1-2t_3z_1^3)} + (-1)^{(\sigma/2)}2\frac{z_3^{2-\sigma/2}z_2^{1+\sigma/2}}{(z_2-z_3)(z_2+z_3)^2}\mathrm{e}^{(t_3z_2^3+tz_2^2+xz_2+t_3z_3^3-tz_3^2+xz_3)} + \frac{z_3^{2-\sigma/2}z_2^{1+\sigma/2}(z_1+z_2)(z_1+z_3)}{(z_2-z_3)(z_2+z_3)^2(z_1-z_2)(z_1-z_3)}\mathrm{e}^{(-2xz_1-2t_3z_1^3+tz_2^2+xz_2+t_3z_2^3-tz_3^2+xz_3+t_3z_3^3)} \tag{66}$$

$$r = \sqrt{2}z_3\mathrm{e}^{(-tz_3^2+xz_3+t_3z_3^3)}\left(1 + \frac{1}{2}\frac{\mathrm{e}^{(-2xz_1-2t_3z_1^3)}(z_1+z_3)}{z_1-z_3}\right)\Big/\tau_2^{(0)} \tag{67}$$

$$q = \sqrt{2}z_2\mathrm{e}^{(tz_2^2+xz_2+t_3z_2^3)}\left(1 - \frac{1}{2}\frac{\mathrm{e}^{(-2xz_1-2t_3z_1^3)}(z_1+z_2)}{z_1-z_2}\right)\Big/\tau_0^{(0)}. \tag{68}$$

In the above examples U_2 and ν can be obtained from (54). We note that we only kept the explicit time dependence on times t_n with $n \leqslant 3$, which was enough to verify the evolution equations (57)–(61).

The characteristic feature of the above soliton solutions is that they mix exponentials $\exp(\sum_{n=1}^{\infty} t_n z_j^n)$ which represent a typical time dependence for the KP solutions with pure KdV-like time dependence of the type $\exp(\sum_{n=0}^{\infty} t_{2n+1} z_j^{2n+1})$ involving only odd times.

The exception is provided by the pure KP type of solution in equations (64), (65), which can also be obtained as a Wronskian arising from the Darboux–Bäcklund transformations.

Our soliton solutions in equations (66)–(68) do not coincide with typical Wronskian soliton expressions valid for the constrained KP hierarchy. The relevant procedures to obtain such soliton solutions for these class of models are derived in [3, 4] using a systematic approach of the Darboux–Bäcklund transformations. This approach obtains the soliton tau function for cKP hierarchy characterized by parameters K and M as a Wronskian:

$$W[f_1, \partial_x^{K+1} f_1, \ldots, \partial_x^{(K+1)N_1} f_1, \ldots, f_2, \partial_x^{K+1} f_2, \ldots, \partial_x^{(K+1)N_2} f_2, \ldots, f_M, \partial_x^{K+1} f_M, \ldots, \partial_x^{(K+1)N_M} f_M]$$

where all the functions f_i with $i = 1, \ldots, M$ satisfy evolution equations:

$$\partial f_i/\partial t_n = \partial_x^n f_i \qquad n = 1, 2, 3, \ldots.$$

The solution of such equation is given by (up to some constants) $f_i = \exp(\sum_{j=1}^{\infty} z_i^j t_j)$. Inserting these functions back into the Wronskian we clearly see that the Darboux–Bäcklund solutions cannot have the same time dependence as the tau-function found in equation (66).

5. Conclusions

In this paper we have given soliton solutions for a cKP hierarchy. A special class of our solutions has not, to the best of our knowledge, previously appeared in the literature. They exhibit a non-trivial mixing of KP times t_n with all indices n's and KdV-like times t_{2n-1} with only odd indices. From the point of view of intermediate gradation, such a mixture is very natural. In fact, we are able to obtain pure KdV solutions, pure KP solutions (meaning all times) and the arbitrary mixtures thereof, just by varying the constant group elements h of the dressing orbit.

As we have seen, these solutions do not fit into the conventional Darboux–Bäcklund method which involves the Wronskian representation. Recall, that such a Wronskian is given in terms of the eigenfunctions f_i satisfying $\partial_n f_i = \partial_x^n f_i$ for all n. This type of time dependence only fits a subclass of our soliton solutions, namely those which are of the pure KP type (meaning that all t_n's with all n's are present without the non-trivial mixture with KdV times). Based on studying many examples of these solutions, we believe that the Wronskian method is able to reproduce that particular sub-class of our solutions.

It may be interesting to study other methods to investigate whether they can be adapted to accomodate the class of the soliton solutions presented above.

Acknowledgments

HA was supported in part by the US Department of Energy grant no DE-FG02-84ER40173. AHZ would like to thank UIC and FAPESP for their hospitality and financial support. LAF, JFG and AHZ acknowledge partial financial support from CNPq.

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